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Reduced-Order Modeling of Unsteady Flows Without Static Correction Requirement

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A new reduced-order modeling approach is presented. This approach is based on fluid eigenmodes and without using the static correction. The vortex lattice method is used to analyze unsteady flows over two-dimensional airfoils and three-dimensional wings. Eigenanalysis and reduced-order modeling are performed using a conventional method with and without the static correction technique. In addition to the conventional method, eigenanalysis and reduced-order modeling are also performed using the new proposed method, that is, without static correction requirement. Numerical examples are presented to demonstrate the accuracy and computational efficiency of the proposed method. Based on the obtained results, it is shown that the accurate reduced-order models of unsteady flows can be constructed without using the static correction technique.

Introduction

REDUCED-ORDER modeling (ROM) is a conceptually novel and computationally efficient technique that has been recently used in analysis of unsteady flows. Unsteady flow eigenmodes are used to construct reduced order unsteady flow models similar to the normal mode analysis in structural dynamics. Although the modal analysis of structures is quite routine, the modal analysis of unsteady flows is still in the developing stage. The advantage of a modal approach is that one may construct a reduced order model by retaining only a few of the original modes. Eigenanalysis of unsteady potential flows about flat airfoils, cascades, and wings have been applied by Hall.[1] He constructed reduced-order models based on an unsteady incompressible vortex lattice method and found that, to obtain satisfactory results, the static correction technique must be used. Florea and Hall[2] also created reduced order models in the time domain for linearized potential flow about airfoils. ROM is used for aerodynamic modeling of the helicopter blades[3] using the Peters finite state airloads model[4] and using nonlinear aeroelastic systems.[5] Romanowski and Dowell[6] applied ROM to subsonic unsteady flows based on the Euler equations about a NACA 0012 airfoil. ROM of unsteady viscous flow in a compressor cascade based on the coupled potential flow and boundary-layer approximation has been applied by Florea et al.,[7] and the status of ROM of unsteady aerodynamic systems has been reviewed by Dowell et al.[8] and Dowell.[9]

Esfahanian and Behbahani-Nejad[10] applied ROM to the subsonic unsteady flows about complex configurations using a boundary element method. They indicated that the number of zero eigenvalues of the unsteady model is equal to the number of elements that lie on the body (Behbahani-Nejad[11]). Hence, some of the eigenmodes that are equal to the body's elements behave exactly in quasi-static fashion, and ROM without the static correction can not generate satisfactory results even with the large number of eigenmodes. On the other hand, ROM based on a body and its wake eigenmodes (conventional ROM) can give satisfactory results if and only if the static correction technique is applied. However, when the static correction technique is applied, the quasi-steady part of the solution must be computed for each time step, which alters the efficiency of ROM. When a reduced-order model is constructed based only on the wake eigenmodes, the body quasi-static eigenmodes are removed, and satisfactory results will be obtained without using the static correction technique.

In this context, a new formulation based on a vortex lattice method is presented by which the eigenvalue problem is defined based only on the unknown wake vortices. The main idea is to present a new ROM that does not require the static correction, or in other words, does not require computation of the quasi steady part at each time step and also has the capabilities of the conventional ROM that requires static correction. The eigenvalues of the new eigensystem are nonzero, and therefore, the new system has no quasi-static eigenmodes. Eigenanalysis results show that the eigenvalues of the proposed method are equal to the corresponding nonzero eigenvalues of the conventional method. To demonstrate the approach, reduced-order models are constructed for unsteady flows over a two-dimensional airfoil and a three-dimensional wing. The results show that the present ROM can accurately and more efficiently analyze unsteady flows in comparison with the conventional reduced-order models.

Eigenanalysis and ROM

In the vortex lattice method (VLM) for unsteady flow computations (UVLM), the following matrix equation can be obtained[1]:

$$\boldsymbol{A}\boldsymbol{\Gamma}^{n+1} + \boldsymbol{B}\boldsymbol{\Gamma}^{n} = \boldsymbol{w}^{n+1} \tag{1}$$

where $\boldsymbol{\Gamma}$ is the vector of vortex strengths, $\boldsymbol{w}$ is the known downwash, and $\boldsymbol{A}$ and $\boldsymbol{B}$ are known sparce matrices. When $\boldsymbol{\Gamma}$ is computed from Eq. (1), unsteady lift can be calculated as[12]

$$L = \int_{-b}^{b} \rho \left[U\gamma(x) + \frac{\mathrm{d}}{\mathrm{d}t} \int_{-b}^{x} \gamma(x_1)\,\mathrm{d}x_1 \right] \mathrm{d}x \tag{2}$$

where $\gamma(x) = \mathrm{d}\Gamma/\mathrm{d}x$ and b is semichord of the airfoil. For zero downwash, one can set $\boldsymbol{\Gamma} = x_i \exp \lambda_i t$ and $z_i = \exp \lambda_i \Delta t$ to obtain

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the following generalized eigenvalue problem:

$$z_i \boldsymbol{A} x_i + \boldsymbol{B} x_i = 0 \tag{3}$$

where λ_i and z_i are ith eigenvalues in λ plane and z plane, respectively, and x_i is the corresponding eigenvector. More generally Eq. (3) can be written as

$$\boldsymbol{AXZ} + \boldsymbol{BX} = \boldsymbol{0} \tag{4}$$

where $\boldsymbol{Z}$ is a diagonal matrix containing the eigenvalues and $\boldsymbol{X}$ is a matrix with columns that are the right eigenvectors. On the other hand, the left eigenvectors satisfy the following relation:

$$\boldsymbol{A}^T \boldsymbol{YZ} + \boldsymbol{B}^T \boldsymbol{Y} = \boldsymbol{0} \tag{5}$$

where $\boldsymbol{Y}$ is a matrix with rows that are the left eigenvectors. If the eigenvectors are normalized suitable, they satisfy the orthogonality conditions

$$\boldsymbol{Y}^T \boldsymbol{AX} = \boldsymbol{I} \tag{6}$$

$$\boldsymbol{Y}^T \boldsymbol{BX} = -\boldsymbol{Z} \tag{7}$$

The dynamic behavior of the fluid can be represented as the sum of the individual eigenmodes, that is,

$$\boldsymbol{\Gamma} = \boldsymbol{Xc} \tag{8}$$

where $\boldsymbol{c}$ is the vector of normal mode coordinates. Substitution of Eq. (8) into Eq. (1), premultiplying by $\boldsymbol{Y}^T$, and making use of the orthogonality condition gives a set of N uncoupled equations for the modal coordinates $\boldsymbol{c}$,

$$\boldsymbol{c}^{n+1} - \boldsymbol{Zc}^n = \boldsymbol{Y}^T \boldsymbol{w}^{n+1} \tag{9}$$

Now one may construct a reduced-order model by retaining only a few of the original modes. However, the preceding reduced-order model does not produce satisfactory results unless the static correction technique is applied. Therefore, it is a normal procedure to decompose the unsteady solution into two parts. One part is equivalent to the response of the system if the disturbance is quasi-steady, and the other part is the dynamic part. Therefore, the unsteady solution can be defined as the following equation:

$$\boldsymbol{\Gamma}^n = \boldsymbol{\Gamma}_s^n + \tilde{\boldsymbol{\Gamma}}^n \tag{10}$$

$$\boldsymbol{\Gamma}^n = \boldsymbol{\Gamma}_s^n + \boldsymbol{X}\tilde{\boldsymbol{c}}^n \tag{11}$$

The quasi-static portion $\boldsymbol{\Gamma}_s$ is given by

$$[\boldsymbol{A} + \boldsymbol{B}]\boldsymbol{\Gamma}_s^n = \boldsymbol{w}^n \tag{12}$$

Thus, Eq. (9) is replaced by

$$\tilde{\boldsymbol{c}}^{n+1} - \boldsymbol{Z}\tilde{\boldsymbol{c}}^n = \boldsymbol{Y}^T \boldsymbol{w}^{n+1} - \boldsymbol{Y}^T\left(\boldsymbol{A}\boldsymbol{\Gamma}_s^{n+1} + \boldsymbol{B}\boldsymbol{\Gamma}_s^n\right) \tag{13}$$

ROM Without Static Correction Requirement

Pervious works[10] show that the existence of zero eigenvalues in the eigensystem is the main reason for applying static correction technique. Here, another approach is proposed that removes zero eigenvalues by defining a new eigenvalue problem that its eigenvalues are the same as the nonzero eigenvalues of the previous eigensystem. If $\boldsymbol{\Gamma}_b$ and $\boldsymbol{\Gamma}_w$ are defined as the vector of a body's and wake vortex strengths, respectively, one can write

$$\boldsymbol{\Gamma} = \begin{Bmatrix} \boldsymbol{\Gamma}_b \\ \boldsymbol{\Gamma}_w \end{Bmatrix} \tag{14}$$

Now, Eq. (1) can be rewritten as

$$\begin{bmatrix} \boldsymbol{A}_{11}\boldsymbol{A}_{12} \\ \boldsymbol{A}_{21}\boldsymbol{A}_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Gamma}_b \\ \boldsymbol{\Gamma}_w \end{Bmatrix}^{n+1} + \begin{bmatrix} \boldsymbol{B}_{11}\boldsymbol{B}_{12} \\ \boldsymbol{B}_{21}\boldsymbol{B}_{22} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\Gamma}_b \\ \boldsymbol{\Gamma}_w \end{Bmatrix}^{n} = \begin{Bmatrix} \boldsymbol{w}_b \\ \boldsymbol{0} \end{Bmatrix}^{n+1} \tag{15}$$

and, therefore,

$$\boldsymbol{A}_{11}\boldsymbol{\Gamma}_b^{n+1} + \boldsymbol{A}_{12}\boldsymbol{\Gamma}_w^{n+1} + \boldsymbol{B}_{11}\boldsymbol{\Gamma}_b^n + \boldsymbol{B}_{12}\boldsymbol{\Gamma}_w^n = \boldsymbol{w}_b^{n+1} \tag{16}$$

$$\boldsymbol{A}_{21}\boldsymbol{\Gamma}_b^{n+1} + \boldsymbol{A}_{22}\boldsymbol{\Gamma}_w^{n+1} + \boldsymbol{B}_{21}\boldsymbol{\Gamma}_b^n + \boldsymbol{B}_{22}\boldsymbol{\Gamma}_w^n = \boldsymbol{0} \tag{17}$$

It can be shown the matrices $\boldsymbol{B}_{11}$ and $\boldsymbol{B}_{12}$ are zero, and therefore, Eq. (17) results in

$$\boldsymbol{w}_b^{n+1} = [\boldsymbol{A}_{11} \quad \boldsymbol{A}_{12}] \begin{Bmatrix} \boldsymbol{\Gamma}_b \\ \boldsymbol{\Gamma}_w \end{Bmatrix}^{n+1} \tag{18}$$

Equation (18) results in

$$\boldsymbol{\Gamma}_b^{n+1} = \boldsymbol{A}_{11}^{-1}\boldsymbol{w}_b^{n+1} - \boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\boldsymbol{\Gamma}_w^{n+1} \tag{19}$$

Substitution of Eq. (19) into Eq. (17) gives

$$\left[\boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\right]\boldsymbol{\Gamma}_w^{n+1} + \left[\boldsymbol{B}_{22} - \boldsymbol{B}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12}\right]\boldsymbol{\Gamma}_w^n = -\boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{w}_b^{n+1} - \boldsymbol{B}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{w}_b^n \tag{20}$$

or

$$\boldsymbol{A}_{\text{new}}\boldsymbol{\Gamma}_w^{n+1} + \boldsymbol{B}_{\text{new}}\boldsymbol{\Gamma}_w^n = \boldsymbol{w}_{\text{new}}^{n+1} \tag{21}$$

where

$$\boldsymbol{A}_{\text{new}} = \boldsymbol{A}_{22} - \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12} \tag{22}$$

$$\boldsymbol{B}_{\text{new}} = \boldsymbol{B}_{22} - \boldsymbol{B}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12} \tag{23}$$

$$\boldsymbol{w}_{\text{new}}^{n+1} = -\boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{w}_b^{n+1} - \boldsymbol{B}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{w}_b^n \tag{24}$$

Because Eq. (21) is vs $\boldsymbol{\Gamma}_w$, the corresponding eigensystem has no zero eigenvalue. Therefore, one may construct accurate reduced-order models without using the static correction technique.

Results and Discussion

Present Vortex Lattice Model

In this section, the results are presented to validate the proposed unsteady vortex lattice model. The unsteady lift due to plunging motion of an isolated airfoil is computed for a range of reduced frequencies and compared with the Theodorsen exact solution as shown in Fig. 1. The airfoil is modeled using 20 vortex elements. The wake length is taken to be 10 chord lengths, and it is discretized using 200 vortex elements. The apparent mass effects have been added to the Theodorsen circulatory lift function to obtain circulatory and noncirculatory lift (see Ref. 12). The results are quite satisfactory for the range of reduced frequencies considered here.

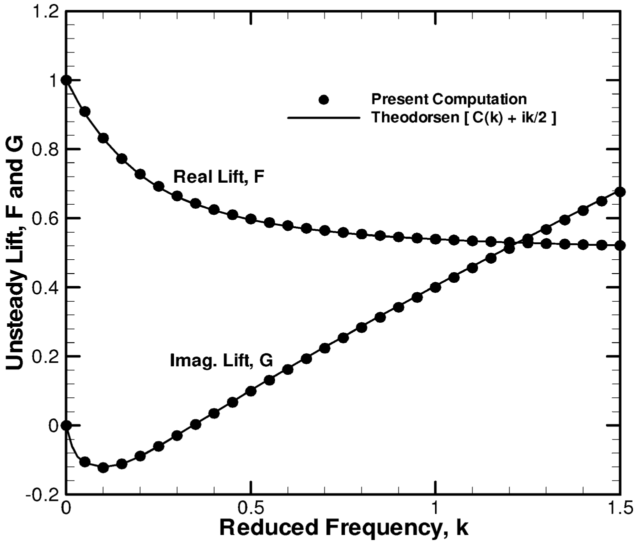

Fig. 1 Unsteady lift due to plunging motion of an isolated airfoil.

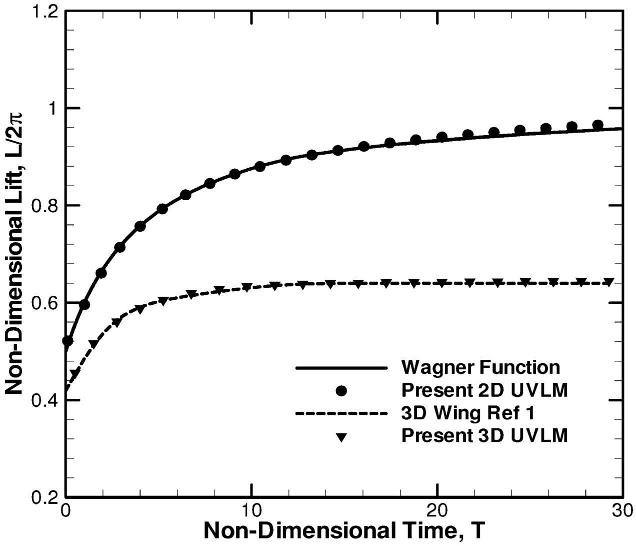

Fig. 2 Lift acting on the airfoil due to step change in airfoil downwash.

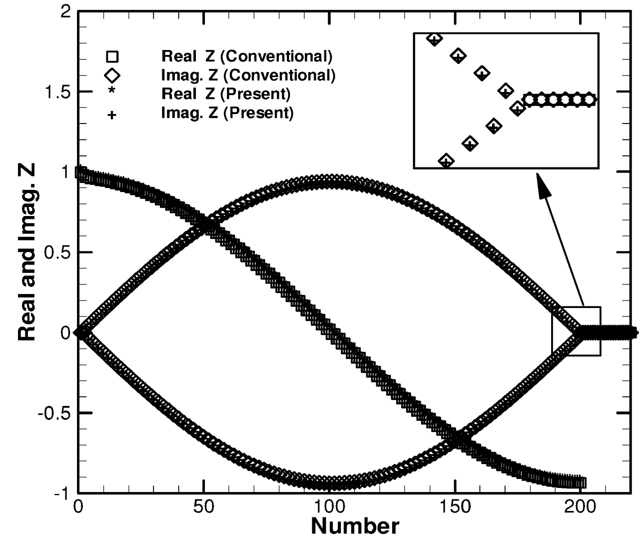

Fig. 4 Eigenvalues vs eigenvalues numbers for two-dimensional airfoil.

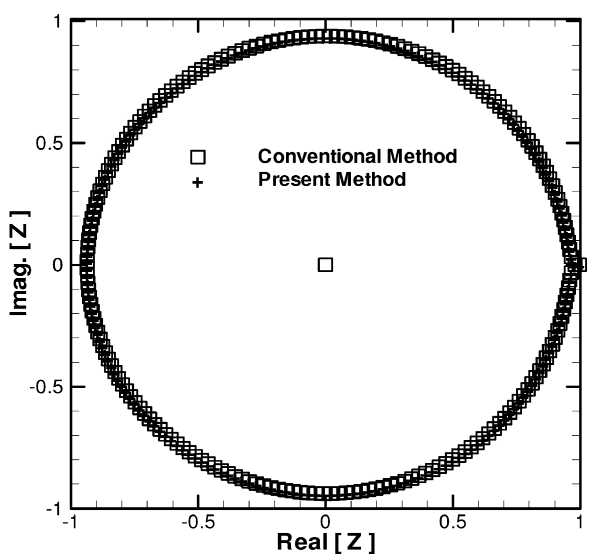

Fig. 3 Eigenvalues for two-dimensional airfoil.

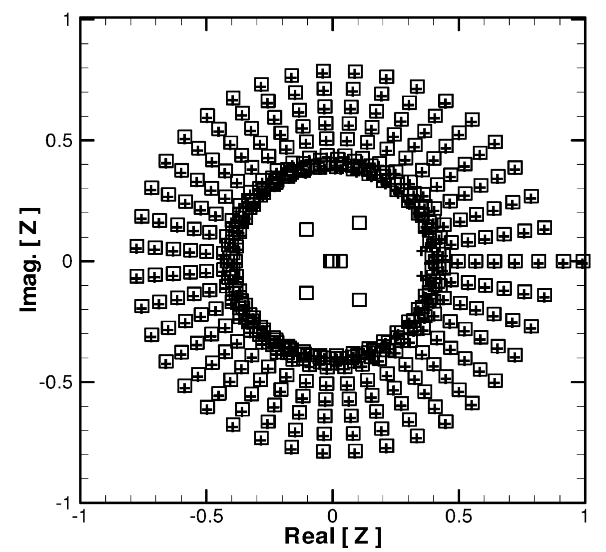

Fig. 5 Eigenvalues of vortex lattice model of unsteady flow about the three-dimensional wing: □, conventional method and +, present method.

The lift acting on the airfoil due to a step change in airfoil downwash (Wagner problem) is considered to validate the present vortex lattice computations in the time domain. As is shown in Fig. 2, the present vortex lattice model results are in perfect agreement with the Wagner function. Moreover, Fig. 2 shows the indicial response of a rectangular wing due to rigid-body plunging motion in comparison with the corresponding results presented in Ref. 1. As in Ref. 1, the wing aspect ratio is 5.0 and it is modeled with 8 vortex elements in the streamwise direction, and 10 in the spanwise direction. The wake is taken to be 5 chords long and is modeled using 40 vortex elements in the streamwise direction and 10 in spanwise direction. As shown Fig. 2, the results of the present three-dimensional vortex lattice model are in good agreement with those of Ref. 1.

Eigenanalysis

The results of conventional and present eigenanalysis are presented in this section. Eigenvalues of the proposed method are shown in Fig. 3 in comparison with those of the conventional method for two-dimensional airfoil. In Fig. 4, the eigenvalues of the proposed method and conventional method are plotted with respect to the eigenvalues numbers. The results show that the eigenvalues of the proposed method are the same as the nonzero eigenvalues of the conventional method. In the proposed eigenanalysis, the eigensystem is interpreted only by the wake elements. Therefore, there are not any zero eigenvalues related to the body's elements. On the other hand, in the conventional method, the eigensystem is constructed using the airfoil elements, as well as the wake elements. Therefore, there are 20 zero eigenvalues related to the body's elements and 200 nonzero eigenvalues related to the wake elements.

Eigenvalues of vortex lattice model of unsteady flow about the three-dimensional wing are plotted in Fig. 5. Again, as is shown in Fig. 5, the nonzero eigenvalues of conventional eigenanalysis are the same as the eigenvalues of the proposed method. Some of the nonzero eigenvalues in Fig. 5 that do not coincide with the eigenvalues of the proposed method correspond to the wing elements and, indeed, they must be zero, but numerical errors in computational problem made them nonzero eigenvalues.

Reduced-Order Models

Next, we use the eigenvalues computed in the preceding section to construct reduced-order aerodynamic models. First we review the conventional reduced-order models (CROM). Figures 6 and 7 show the unsteady lift predicted using the ROM technique without the static correction for a two-dimensional airfoil and three-dimensional wing, respectively. As expected and shown in Figs. 6 and 7, CROM without the static correction can not produce satisfactory results even if a large number of modes are used. This is because the effects of the eigenmodes corresponding to the zero eigenvalues may be only considered using static correction.

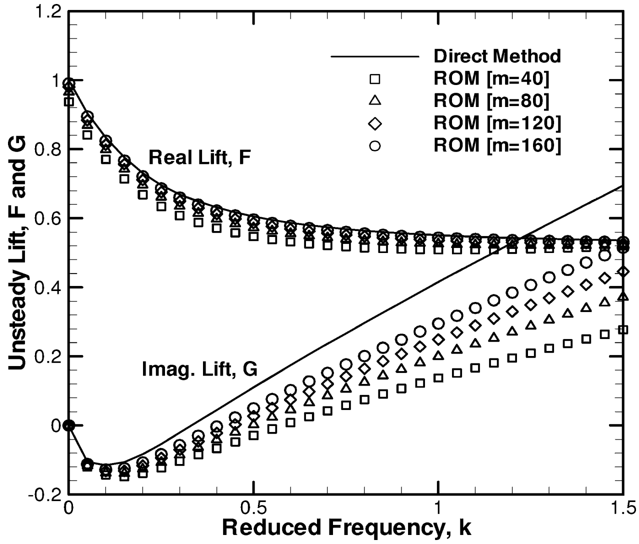

Fig. 6 Unsteady lift for two-dimensional airfoil predicted using ROM without static correction.

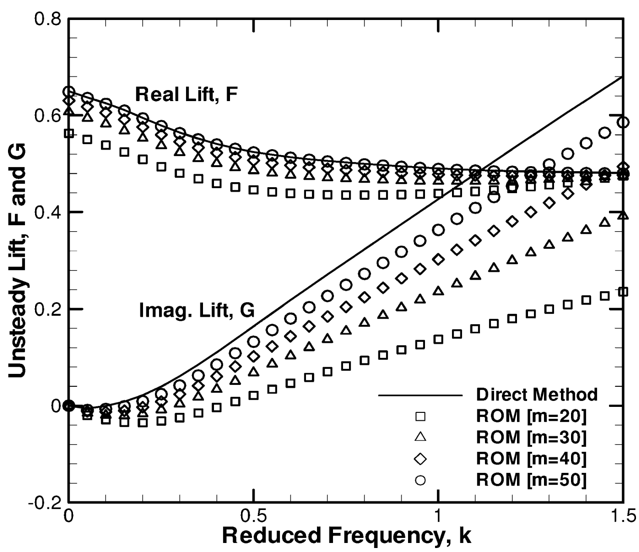

Fig. 7 Unsteady lift for three-dimensional wing predicted using ROM without static correction.

The accuracy of the proposed method is shown in Fig. 8, where the unsteady lift of the airfoil predicted using the CROM with static correction and the present reduced-order models (PROM) are compared. As is shown in Fig. 8, PROM without static correction can produce satisfactory results, as accurate as CROM with the static correction technique. This is due to the absence of zero eigenvalues in the new eigensystem. The number of modes effects on the unsteady lift for a two-dimensional airfoil is shown in Fig. 9. It can be seen that, by increasing the number of modes, the difference between direct method and PROM decreases. The unsteady lift of the three-dimensional wing predicted using CROM with static correction and PROM are compared in Fig. 10. Figure 10 shows that the same accuracy will be obtained for the three-dimensional wing. The efficiency of the proposed method will be more clear if the method is used in the time-domain analysis. CROM and PROM are used in the time domain to predict the unsteady lift due to plunging motion with reduced frequency $k = 0.5$. Computational results of the lift variation during some heaving oscillation cycles of the airfoil are presented in Fig. 11. The results of the PROM and CROM with static correction are in perfect agreement with those of the direct method. However, the results of CROM without the static correction show considerable error, which is expected. Figure 12 presents the same comparison for the three-dimensional wing. Similarly, the

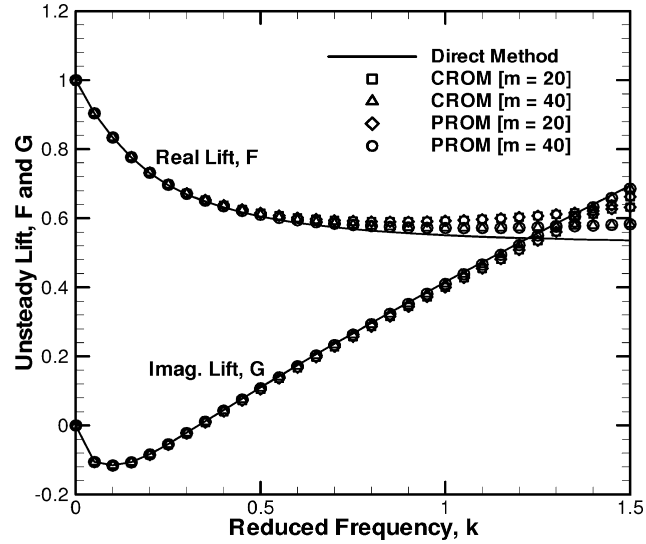

Fig. 8 Unsteady lift for two-dimensional airfoil predicted using CROM with static correction and PROM.

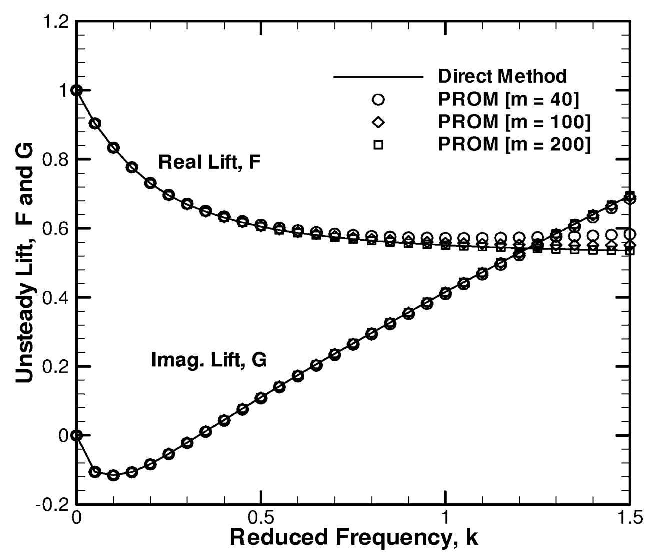

Fig. 9 Number of modes effects on unsteady lift for two-dimensional airfoil.

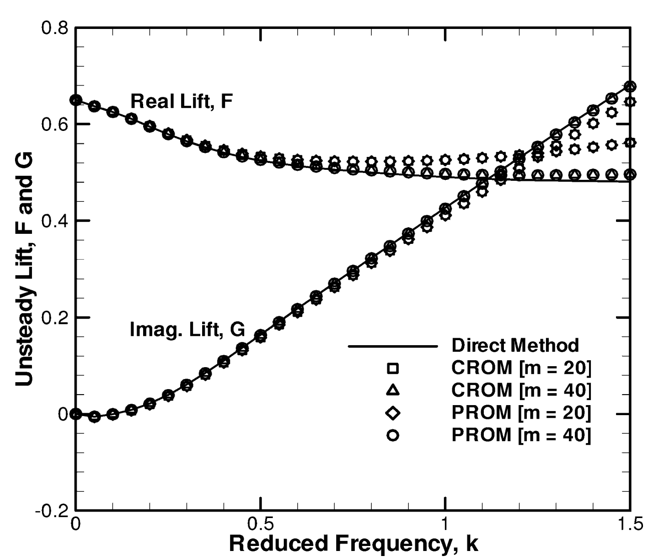

Fig. 10 Unsteady lift for three-dimensional wing predicted using CROM with static correction and PROM.

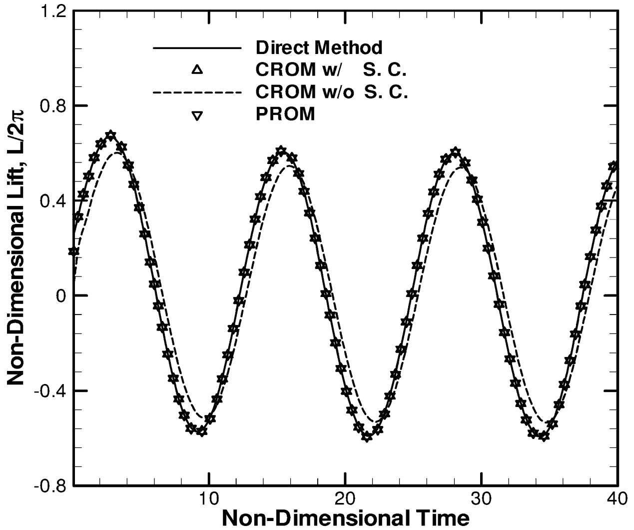

Fig. 11 Lift variation during some heaving oscillation cycles of two-dimensional airfoil.

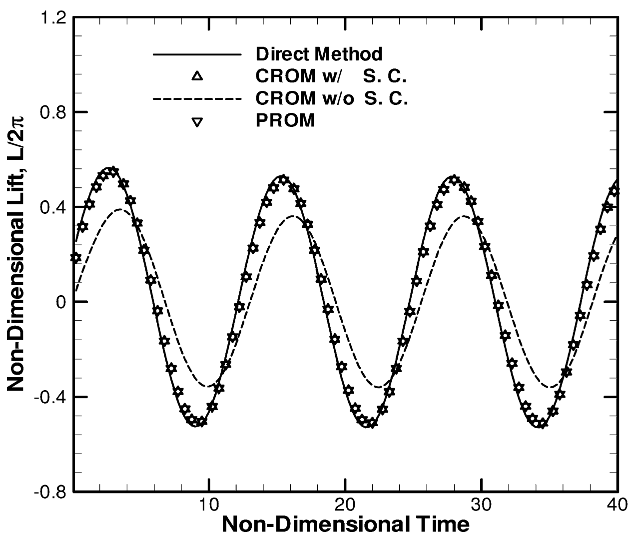

Fig. 12 Lift variation during some heaving oscillation cycles of three-dimensional wing.

results of PROM show the same accuracy as CPROM with the static correction.

Efficiency Analysis

Finally, the efficiency of PROM is presented for both the frequency and the time domains. To clarify the efficiency analysis, CPU times for PROM and CROM with static correction are compared. In addition, CPU times for eigenvalue computations and ROM are presented separately. Table 1 indicates CPU times in seconds for the two-dimensional airfoil and the three-dimensional wing. For each case, ROM is performed in the time domain (TD) and in the frequency domain (FD). The results are based on numerical computations using a P3-1000 MHz with 1-GB RAM. The results presented in Table 1 reveal that the proposed method can analyze either eigensystem or ROM more efficiently than CROM with the static correction. Efficiency of the proposed method from the eigenanalysis point of view is due to fact that the resulting eigensystem has a smaller rank than the conventional method inasmuch as it is represented based only on the wake elements. Therefore, the proposed method will be more efficient when the ratio of number of body elements to the number of wake elements is increased. The application of the proposed method for ROM is more efficient than CROM because there is no need to compute the quasi-steady solution in each time step. Hence, the present method will be more efficient as the time increases in the TD analysis.

Table 1 CPU time (seconds) comparison between conventional and present methods

	Eigenanalysis		ROM	
Case	Conventional	Present	Conventional	Present
Two dimensional				
TD	10	8	7.9	5.7
FD	10	8	0.7	0.2
Three dimensional				
TD	119	79	12.5	4.9
FD	119	79	3.8	2.2

Conclusions

Conventional ROM can generate satisfactory results when the static correction is used. This effect is due to the exitance of zero eigenvalues in the z plane. It is shown that the existence of zero eigenvalues depends on the number of computational elements of the body. For implementation of static correction, one needs to find quasi-steady solution in each time step. In the present work, a reduced-order model for unsteady flows is developed without need of the static correction technique. In this method, the numerical eigensystem is constructed using the wake variables only, and the rank of the eigensystem is lower than the corresponding eigensystem of the conventional method. Therefore, the eigensystem in the z plane does not contain any zero eigenvalues and there is no need for the static correction. The results indicate that the proposed method is computationally more efficient than the conventional method, which requires the static correction. Moreover, the obtained results indicate that the proposed method can produce satisfactory results as accurate as the conventional method.

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